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HAMILTON'S EQUATIONS OF GEOMETRIC OPTICS[1]

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Abstract

Starting with a general dispersion equation, the expression $\partial\omega/\partial\vec{k}$ for the group velocity, and the refraction laws of Sommerfeld-Runge and Poeverlein, Hamilton's equations for ray tracing and group speed are deduced here in vector notation (the consequent use of which simplifies also the introduction of curvilinear coordinates). Some particular forms of the dispersion equation are discussed, especially for electromagnetic waves in an incompressible plasma pervaded by a magnetic field.

1. Dispersion equation

Since the pioneering paper of Haselgrove [1955] about ray tracing in anisotropic media this has become an important tool for the theoretical explanation of measurements where the propagation of electromagnetic waves in the ionospheric or magnetospheric plasma is involved and the influence of the earth's magnetic field $\vec{B}$ is important. The latter causes the anisotropy of the plasma resulting in a complicated form of the dispersion equation

$$F(\vec{k},\omega;\vec{r},t) = 0 \quad . \tag{1}$$

For an incompressible ("cold") plasma this dispersion equation can be written as a quadratic equation in k^2 [Stix 1962]

$$Ak^4 - B\left(\frac{\omega}{c_o}\right)^2 k^2 + \left(\frac{\omega}{c_o}\right)^4 C = 0 \quad . \tag{2}$$

The coefficients A, B, C comprise the three eigenvalues $\check{\varepsilon}_s$ $(s = -1, 0, +1)$ of the effective dielectric tensor

$$\check{\overset{\leftrightarrow}{\varepsilon}} \equiv \varepsilon_o \overset{\leftrightarrow}{U} + \frac{i}{\omega}\overset{\leftrightarrow}{\sigma} \qquad \begin{array}{l} \overset{\leftrightarrow}{U} \equiv \text{unit tensor} \\ \overset{\leftrightarrow}{\sigma} \equiv \text{conductivity tensor} \end{array}$$

viz.

$$\check{\varepsilon}_s = \varepsilon_o - \sum_c \frac{q_c^2 N_c/m_c\, \omega}{\omega + i\nu_c + sq_c B/m_c c_o\sqrt{\varepsilon_o\mu_o}} \qquad s = -1,0,+1 \tag{3}$$

and the angle θ between the propagation vector $\vec{k}$ and the earth's magnetic field $\vec{B}$:

$$\cos\theta \equiv \hat{\vec{k}} \cdot \hat{\vec{B}} = \frac{\vec{k}\cdot\vec{B}}{kB} = \frac{\vec{k}\cdot\vec{B}}{\sqrt{\vec{k}\cdot\vec{k}}\sqrt{\vec{B}\cdot\vec{B}}} \quad . \tag{4}$$

The eigenvalues $\check{\varepsilon}_s$ (3) and their linear combinations

$$\check{\varepsilon}_{\pm} \equiv \frac{1}{2}(\check{\varepsilon}_{+1} \pm \check{\varepsilon}_{-1})$$

depend on the wave frequency $\omega/2\pi$ and on $\vec{r},t$ *via* the densities N_c, the collision frequencies ν_c, and $|\vec{B}|$. Hence A , B , C are functions of $\cos\theta$, ω and $\vec{r},t$:

$$A \equiv \frac{\check{\varepsilon}_+}{\varepsilon_o} + \frac{\check{\varepsilon}_o - \check{\varepsilon}_+}{\varepsilon_o}\cos^2\theta$$

$$B \equiv \frac{\check{\varepsilon}_o\,\check{\varepsilon}_+ + \check{\varepsilon}_{+1}\check{\varepsilon}_{-1}}{\varepsilon_o^2} + \frac{\check{\varepsilon}_o\,\check{\varepsilon}_+ - \check{\varepsilon}_{+1}\,\check{\varepsilon}_{-1}}{\varepsilon_o^2}\cos^2\theta \tag{5}$$

$$C \equiv \frac{\check{\varepsilon}_o\,\check{\varepsilon}_{+1}\,\check{\varepsilon}_{-1}}{\varepsilon_o^3}$$

For wave pulsations ω very large in comparison with the gyro pulsations $q_i\, B/m_i c_o\sqrt{\varepsilon_o\mu_o}$ of the ions the electrons alone need to be considered. With the standard notation

$$\tilde{X} \equiv \frac{X}{1+iZ} \equiv \frac{q_e^2 N_e/\varepsilon_o m_e \omega^2}{1+i\nu_e/\omega}$$

$$\tilde{Y} \equiv \frac{Y}{1+iZ} \equiv \frac{|q_e| B/m_e c_o \sqrt{\varepsilon_o \mu_o}\,\omega}{1+i\nu_e/\omega}$$

the coefficients A , B , C can be written

$$A(1 - \tilde{Y}^2) = 1 - \tilde{X} - \tilde{Y}^2 + \tilde{X}\tilde{Y}^2 \cos^2\theta$$

$$B(1 - \tilde{Y}^2) = 2(1 - \tilde{X})^2 - (2 - \tilde{X})\tilde{Y}^2 + \tilde{X}\tilde{Y}^2 \cos^2\theta \qquad (6)$$

$$C(1 - \tilde{Y}^2) = (1 - \tilde{X})^3 - (1 - \tilde{X})\tilde{Y}^2$$

and the solution of the dispersion equation (2) is the well-known dispersion formula of Appleton [1928] and Lassen [1927] .

The taking into account of the collision frequencies ν_c gives rise to complex coefficients (5) in the dispersion equation (2). By separating its real and imaginary parts we obtain two coupled equations for the (real) propagation vector Re $\vec{k}$ and the absorption vector Im $\vec{k}$. For their solution we have to prescribe both the directions of Re $\vec{k}$ and of Im $\vec{k}$. (They coincide only for vertical incidence of waves into a "stratified" ionosphere.) This makes

$$\cos\theta \equiv \frac{\mathrm{Re}\,\vec{k} + i\,\mathrm{Im}\,\vec{k}}{\sqrt{(\mathrm{Re}\,\vec{k})^2 - (\mathrm{Im}\,\vec{k})^2 + 2i\,\mathrm{Re}\,\vec{k}\cdot\mathrm{Im}\,\vec{k}}} \cdot \hat{\vec{B}} \qquad [4]$$

in general complex and thus complicates considerably all calculations. Only with

$$|\mathrm{Im}\,\vec{k}| \ll |\mathrm{Re}\,\vec{k}| \quad \text{for weak absorption} \tag{7}$$

we may regard

$$\cos\theta \approx \frac{\mathrm{Re}\,\vec{k}}{|\mathrm{Re}\,\vec{k}|} \cdot \hat{\vec{B}} \qquad [4]$$

as (nearly) real and

$$0 = \mathrm{Re}\,F(\mathrm{Re}\,\vec{k} + i\,\mathrm{Im}\,\vec{k},\ \omega; r, t) \approx \mathrm{Re}\,F(\mathrm{Re}\,\vec{k},\ \omega; \vec{r}, t) \qquad [1]$$

as the dispersion equation to be solved for a given direction of $\mathrm{Re}\,\vec{k}$.

For the sake of simplicity we neglect the absorption completely in the following. Weak absorption (7) can be taken into account by writing $\mathrm{Re}\,F$ and $\mathrm{Re}\,\vec{k}$ instead of F and $\vec{k}$ in the finite results.

2. Hamilton's equations

For the calculation of group propagation we start with the well-known expression

$$\frac{d\vec{r}}{dt} = \frac{\partial\omega}{\partial\vec{k}} \tag{8}$$

for the group velocity. Since the dispersion equation (1) gives only a connection between ω and k and requires a given $\hat{\vec{k}}$ we need further equations for the determination of $\hat{\vec{k}}$ during the group travel. These are given by the refraction law of Sommerfeld and Runge [1911] and its four-dimensional generalization by Poeverlein [1962] :

$$\frac{\partial}{\partial \vec{r}} \times \vec{k} = 0 \ , \qquad \frac{\partial \vec{k}}{\partial t} + \frac{\partial \omega}{\partial \vec{r}} = 0 \ . \tag{9}$$

Together with the dispersion equation (1) the equations (8) (9) suffice for the caluclation of the group propagation. But they are rather unhandy for an actual computation and therefore we will transform them into a more convenient form.

For this we differentiate the dispersion equation

$$F(\vec{k},\omega;\vec{r},t) = 0 \tag{1}$$

totally with respect to $\vec{r}$:

$$0 = \frac{dF}{d\vec{r}} = \frac{\partial F}{\partial \vec{r}} + \left(\frac{\partial}{\partial \vec{r}} \ \vec{k}\right) \cdot \frac{\partial F}{\partial \vec{k}} + \frac{\partial \omega}{\partial \vec{r}} \ \frac{\partial F}{\partial \omega} \ .$$

Combining this equation with the Somerfeld-Runge law (9) in the form

$$0 = \frac{\partial F}{\partial \vec{k}} \times \left(\frac{\partial}{\partial \vec{r}} \times \vec{k}\right) = \left(\frac{\partial}{\partial \vec{r}} \ \vec{k}\right) \cdot \frac{\partial F}{\partial \vec{k}} - \frac{\partial F}{\partial \vec{k}} \cdot \frac{\partial}{\partial \vec{r}} \ \vec{k}$$

and with Poeverlein's law (9) we obtain

$$0 = \frac{\partial F}{\partial \vec{r}} + \frac{\partial F}{\partial \vec{k}} \cdot \frac{\partial}{\partial \vec{r}} \vec{k} - \frac{\partial \vec{k}}{\partial t} \frac{\partial F}{\partial \omega} = \frac{\partial F}{\partial \vec{r}} - \frac{\partial F}{\partial \omega} \left(\frac{\partial}{\partial t} - \frac{\partial F/\partial \vec{k}}{\partial F/\partial \omega} \cdot \frac{\partial}{\partial \vec{r}} \right) \vec{k} \, . \tag{10}$$

Total differentiation of the dispersion equation (1) with respect to $\vec{k}$ yields

$$0 = \frac{dF}{d\vec{k}} = \frac{\partial F}{\partial \vec{k}} + \frac{\partial \omega}{\partial \vec{k}} \; \frac{\partial F}{\partial \omega}$$

and hence with Eq. (8) the expression

$$\frac{d\vec{r}}{dt} = - \frac{\partial F}{\partial \vec{k}} \Big/ \frac{\partial F}{\partial \omega} \tag{11}$$

for the group velocity. We now recognize the parenthesis in Eq. (10) as the "mobile operator"

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{d\vec{r}}{dt} \cdot \frac{\partial}{\partial \vec{r}}$$

describing the total temporal variation at a point traveling with $d\vec{r}/dt$. Therefore Eq. (10) can be written

$$\frac{d\vec{k}}{dt} = \frac{\partial F}{\partial \vec{r}} \; \frac{\partial F}{\partial \omega} \quad . \tag{12}$$

In a completely analogous manner the total temporal differentiation of the dispersion equation (1) leads to

$$\frac{d\omega}{dt} = - \frac{\partial F}{\partial t} \Big/ \frac{\partial F}{\partial \omega} \ . \tag{13}$$

The three equations

$$\frac{d\vec{r}}{dt} = - \frac{\partial F}{\partial \vec{k}} \Big/ \frac{\partial F}{\partial \omega} \ , \quad \frac{d\vec{k}}{dt} = \frac{\partial F}{\partial \vec{r}} \Big/ \frac{\partial F}{\partial \omega} \ , \quad \frac{d\omega}{dt} = - \frac{\partial F}{\partial t} \Big/ \frac{\partial F}{\partial \omega} \qquad [11] [12] [13]$$

together with the dispersion equation (1) - are now sufficient for the calculation of the group propagation. Since they have the same denominator we can write them in a more symmetrical form by introducing a differential parameter $d\tau$ which describes the propagation along the group path (ray):

$$\frac{d\vec{r}}{d\tau} = \frac{\partial F}{\partial \vec{k}} \qquad \frac{d\vec{k}}{d\tau} = - \frac{\partial F}{\partial \vec{r}} \tag{14a}$$

$$\frac{dt}{d\tau} = - \frac{\partial F}{\partial \omega} \qquad \frac{d\omega}{d\tau} = \frac{\partial F}{\partial t} \tag{14b}$$

The vector equations (14a) are Hamilton's equations for geometrical optics, the two scalar equations (14b) their four-dimensional generalizations [Synge 1954] . The parameter $d\tau$ depends on the particular form of the dispersion equation (1). Its elimination leads back to the equations (11) (12) (13); but for a numerical solution of the equations (14) it is preferable to retain $d\tau$.

For physical interpretations it is sometimes useful to have the group path element

$$ds = \sqrt{d\vec{r} \cdot d\vec{r}}$$

as a differential parameter instead of the merely mathematical quantity $d\tau$. From the first vectorial Hamilton equation (14a) we obtain by scalar multiplication with itself immediately

$$\frac{ds}{d\tau} = \left| \frac{\partial F}{\partial \vec{k}} \right| \tag{15}$$

which we can use to replace $d\tau$ by ds .

3. Particular forms of Hamilton's equations

As was pointed out by Synge [1954] we may use as dispersion equation

$$F(\vec{k},\omega;\vec{r},t) = 0 \qquad [1]$$

not only an equation of the form

$$k^4 A(\omega;\vec{r},t) - k^2 \left(\frac{\omega}{c_o}\right)^2 B(\omega;\vec{r},t) + \left(\frac{\omega}{c_o}\right)^4 C(\omega;\vec{r},t) = 0 \qquad [2]$$

but also any solution of it written in a proper form. In the following we will discuss three particular examples.

If we have a solution of the form $\omega = \omega(\vec{k};\vec{r},t)$ we can write a new dispersion equation (after multiplication with Planck's constant $\hbar$)

$$F_H \equiv \hbar\omega - \hbar\omega\,(\hbar\vec{k};\vec{r},t) = E - H(\vec{p};\vec{r},t) = 0$$

put this into the general Hamilton equations (14) obtaining $dt/d\tau_H = \hbar$ and

$$\frac{d\vec{r}}{dt} = \frac{\partial H}{\partial \vec{p}} \quad , \quad \frac{d\vec{p}}{dt} = -\frac{\partial H}{\partial \vec{r}} \quad , \quad \frac{dE}{dt} = \frac{\partial H}{\partial t} \quad .$$

These are Hamilton's canonical equations in analytical mechanics, *i.e.* the equations for the group propagation of matter waves.

With any of the two solutions

$$k_{\pm}^2(\hat{\vec{k}},\omega;\vec{r},t) = \left(\frac{\omega}{c_o}\right)^2 \frac{B}{2A}\left[1 \mp \sqrt{1 - 4\,\frac{AC}{B^2}}\,\right]$$

of Eq. (2) we can write a dispersion equation

$$F_k \equiv k - k(\hat{\vec{k}},\omega;\vec{r},t) = 0 \tag{16}$$

and find from the first scalar Hamilton equation (14b) together with (8)

$$d\tau_k = \frac{\partial \omega}{\partial k}\,dt = \hat{\vec{k}} \cdot \frac{\partial \omega}{\partial \vec{k}}\,dt = \hat{\vec{k}} \cdot d\vec{r} \quad .$$

This is the longitudinal component of the group path element $d\vec{r}$, *i.e.* its projection upon the wave normal $\hat{\vec{k}}$.

Dividing $F_k = 0$ (16) by $k(\hat{\vec{k}},\omega;\vec{r},t)$ we obtain the dispersion equation used by Haselgrove [1955] :

$$G - 1 \equiv \frac{F_k(\vec{k},\omega;\vec{r},t)}{k(\hat{\vec{k}},\omega;\vec{r},t)} = \frac{k}{k(\hat{\vec{k}},\omega;\vec{r},t)} - 1 = 0$$

which yields

$$d\tau_{G-1} = k\,\frac{\partial\omega}{\partial k}\,dt = \vec{k}\cdot d\vec{r} \quad .$$

4. Coupled vector equation

The second vectorial Hamilton equation (14a) describes the variation of $\vec{k}$ along the ray. Instead of this it is often advantageous to have an equation for the variation of the direction $\hat{\vec{k}}$ alone, because after its solution we can determine the absolute value k by means of the dispersion formula

$$k = k(\hat{\vec{k}},\omega;\vec{r},t) \quad . \qquad [16]$$

Therefore we can decompose the variation of $\vec{k}$ into the variations of k and $\hat{\vec{k}}$.

Doing this in the differential operator $\partial/\partial\vec{k}$ of the first Hamilton equation (14a) we obtain

$$\frac{\partial}{\partial\vec{k}} = \hat{\vec{k}}\,\frac{\partial}{\partial k} - \hat{\vec{k}} \times \left(\hat{\vec{k}} \times \frac{\partial}{\partial\vec{k}}\right) \quad . \qquad (17)$$

The first (longitundinal) term acts only upon k , the second (transversal) one only upon $\hat{\vec{k}}$. (The decomposition (17) can easily be verified by expanding the double vector product.) In our gyrotropic plasma $\hat{\vec{k}}$ appears in the function $F(k,\hat{\vec{k}},\omega)$ only in the combination

$$\cos\theta \equiv \hat{\vec{k}} \cdot \hat{\vec{B}} \tag{4}$$

[see Eq. (5)] and we have

$$-\hat{\vec{k}} \times \left(\hat{\vec{k}} \times \frac{\partial}{\partial \vec{k}}\right) F = -\hat{\vec{k}} \times \left(\hat{\vec{k}} \times \frac{\partial \hat{\vec{k}} \cdot \hat{\vec{B}}}{k\, \partial\, \hat{\vec{k}}}\right) \frac{\partial F}{\partial(\hat{\vec{k}} \cdot \hat{\vec{B}})}$$

$$= -\hat{\vec{k}} \times \left(\hat{\vec{k}} \times \frac{\hat{\vec{B}}}{k}\right) \frac{\partial F}{\partial \cos\theta} = \frac{\hat{\vec{B}} - \hat{\vec{k}}\cos\theta}{k} \, \frac{\partial F}{\partial \cos\theta} \, . \tag{18}$$

With Eqs. (17) (18) the first Hamilton equation (14a) is written

$$\frac{d\vec{r}}{d\tau} = \left[\hat{\vec{k}} \frac{\partial}{\partial k} + \frac{\hat{\vec{B}} - \hat{\vec{k}}\cos\theta}{k} \, \frac{\partial}{\partial \cos\theta} \right] F(k,\cos\theta,\omega;\vec{r},t) \tag{19}$$

Putting $\vec{k} = k\,\hat{\vec{k}}$ into the second Hamilton equation (14a) we obtain

$$k \frac{d\hat{\vec{k}}}{d\tau} + \hat{\vec{k}} \frac{dk}{d\tau} = - \frac{\partial F}{\partial \vec{r}}$$

To eliminate $dk/d\tau$ we multiply this equation scalarly with $\hat{\vec{k}}$ getting (with $\hat{\vec{k}} \cdot d\hat{\vec{k}} = 0$)

$$\frac{dk}{d\tau} = - \hat{\vec{k}} \cdot \frac{\partial F}{\partial \vec{r}}$$

and hence

$$k \frac{d\hat{\vec{k}}}{d\tau} = - \frac{\partial F}{\partial \vec{r}} + \hat{\vec{k}}\, \hat{\vec{k}} \cdot \frac{\partial F}{\partial \vec{r}} \quad . \tag{20}$$

The solution $\hat{\vec{k}}(\omega;\tau(\vec{r}))$ of Eq. (20) gives us the wave normal along the ray $\tau(\vec{r})$, *i.e.* the phase path. With this we solve Eq. (19) for the ray itself. Thus the coupled vector equations (19)(20) together with the dispersion formula (15) form a complete set for the determination of phase path and group path (ray).

To replace the mathematical parameter $d\tau$ by the group path element ds we use

$$\frac{ds}{d\tau} = \left| \frac{\partial F}{\partial \vec{k}} \right| \qquad [15$$

with Eq. (19) and obtain

$$\frac{ds}{d\tau} = \sqrt{\left(\frac{\partial F}{\partial k}\right)^2 + \frac{\sin^2\theta}{k^2} \left(\frac{\partial F}{\partial \cos\theta}\right)^2} \tag{21}$$

Dividing eqs. (19) (20) by eq. (21) we obtain coupled vector equations for the unit vector $d\vec{r}/ds$ and $d\hat{\vec{k}}/ds$. Dividing Eq. (14b) by $ds/d\tau$ we get an equation for the reciprocal group speed

$$\frac{dt}{ds} = -\frac{d\tau}{ds}\frac{\partial F}{\partial \omega} . \tag{22a}$$

Integration along the ray path yields the group travel time

$$t - t_o = -\int_{\tau_o}^{\tau} d\tau \frac{\partial F}{\partial \omega} = -\int_{s_o}^{s} ds \; \frac{d\tau}{ds}\frac{\partial F}{\partial \omega} \tag{22b}$$

5. The coefficients in the first Hamilton equation for gyrotropic incompressible plasma.

The coefficients

$$\frac{\partial F}{\partial k} \quad \text{and} \quad \frac{1}{k}\,\frac{\partial F}{\partial \cos\theta}$$

on the right-hand side of the first Hamilton equation (19), which enter also the expression (21) for $ds/d\tau$, can easily be calculated from the dispersion equation

$$F(\vec{k},\omega;\ \vec{r},t) \equiv k^4 A - k^2\left(\frac{\omega}{c_o}\right)^2 B + \left(\frac{\omega}{c_o}\right)^4 C = 0 \tag*{[2]}$$

with the coefficients A, B, C given by Eqs. (5) or (6). Since F is quadratic in k^2 we have

$$\frac{\partial F}{\partial k_\pm} = 2k_\pm \quad (\pm\sqrt{\text{discriminant of } F}\,) = 2k_\pm\left(\frac{\omega}{c_o}\right)^2 \quad (\pm\sqrt{B^2 - 4AC}\,)$$

which can be written as [Stix 1962]

$$\frac{\partial F}{\partial k_\pm} = 2k_\pm \left(\frac{\omega}{c_o}\right)^2 \frac{\check{\varepsilon}_o \check{\varepsilon}_-}{\varepsilon_o^{\;2}} \; 2\cos\theta \; \frac{\mp\sqrt{1+\zeta^2}}{\zeta} \tag{23}$$

with

$$\frac{\check{\varepsilon}_o \check{\varepsilon}_-}{\varepsilon_o^{\;2}} \approx (1-\tilde{X}) \; \frac{-\tilde{X}\tilde{Y}}{1-\tilde{Y}^2} \tag{23a}$$

and

$$\zeta \equiv \frac{2\cos\theta}{\sin^2\theta} \; \frac{\check{\varepsilon}_o \check{\varepsilon}_-}{\check{\varepsilon}_{+1}\check{\varepsilon}_{-1} - \check{\varepsilon}_o \check{\varepsilon}_+} \approx \frac{2\cos\theta}{\sin^2\theta} \; \frac{1-\tilde{X}}{\check{Y}} \tag{23b}$$

The correspondence of the $\pm$-signs in (23) is conventional in ionospheric physics, *cf.* Eq. (27b) below. The approximate forms of $\check{\varepsilon}_o\check{\varepsilon}_-$ and ζ are valid if only electrons (and no ions) are taken into account.

We write the second coefficient in a similar manner:

$$\frac{1}{k} \; \frac{\partial F}{\partial \cos\theta} = 2k \left(\frac{\omega}{c_o}\right)^2 \frac{\check{\varepsilon}_o \check{\varepsilon}_-}{\varepsilon_o^2} \; \frac{2\cos^2\theta}{\sin^2\theta} \; \frac{1-\left(\frac{c_o}{\omega}k\right)^2 \frac{\zeta}{\delta}}{\zeta} \tag{24}$$

with

$$\delta \equiv \frac{2\cos\theta}{\sin^2\theta} \; \frac{\check{\varepsilon}_o \check{\varepsilon}_-}{\varepsilon_o(\check{\varepsilon}_+ - \check{\varepsilon}_o)} \approx \frac{2\cos\theta}{\sin^2\theta} \; \frac{1-\check{X}}{\tilde{Y}} \tag{24a}$$

The normalization relation (21) for the introduction of the group path element ds is written with Eqs. (23) (24)

$$\left(\frac{ds}{d\tau}\right)_{\pm} = \mp 2k_{\pm}\left(\frac{\omega}{c_o}\right)^2 \frac{\varepsilon_o \varepsilon_-}{\varepsilon_o^2} \frac{2\cos\theta}{\zeta} \sqrt{1+\zeta^2+\cot^2\theta\left[1-\left(\frac{c_o}{\omega}k\right)^2 \frac{\zeta}{\delta}\right]^2} \tag{25}$$

Therefore the first Hamilton equation

$$\frac{d\vec{r}}{d\tau} = \hat{\vec{k}}\,\frac{\partial F}{\partial k} + \frac{\hat{\vec{B}} - \hat{\vec{k}}\cos\theta}{k}\,\frac{\partial F}{\partial \cos\theta} \tag{[19]}$$

can now be written with Eqs. (23) (24) and (25) as an equation for the unit vector $d\vec{r}/ds$:

$$\left(\frac{d\vec{r}}{ds}\right)_{\pm} = \frac{\pm\sqrt{1+\zeta^2}\;\hat{\vec{k}} + \dfrac{\hat{\vec{k}}\cos\theta - \hat{\vec{B}}}{\sin\theta}\cot\theta\left[1-\left(\dfrac{c_o}{\omega}k_{\pm}\right)^2\dfrac{\zeta}{\delta}\right]}{\pm\sqrt{1+\zeta^2+\cot^2\theta\left[1-\left(\dfrac{c_o}{\omega}k_{\pm}\right)^2\dfrac{\zeta}{\delta}\right]^2}} \tag{26}$$

The vector

$$\frac{\hat{\vec{k}}\cos\theta - \hat{\vec{B}}}{|\sin\theta|} = \frac{\hat{\vec{k}}\times(\hat{\vec{k}}\times\hat{\vec{B}})}{|\hat{\vec{k}}\times(\hat{\vec{k}}\times\hat{\vec{B}})|} \tag{26a}$$

is a transversal unit vector (perpendicular to the wave normal $\hat{\vec{k}}$).

6. Discussion of limiting cases

For the (numerical) integration of the second Hamilton equation

$$\frac{d\vec{k}}{ds} = -\frac{d\tau}{ds}\,\frac{\partial F}{\partial \vec{r}} \tag{[14a]}$$

or the equivalent equation

$$\frac{d\hat{\vec{k}}}{ds} = -\frac{1}{k}\frac{d\tau}{ds}\left(\frac{\partial F}{\partial \vec{r}} - \hat{\vec{k}}\,\hat{\vec{k}}\cdot\frac{\partial F}{\partial \vec{r}}\right) \qquad [20]$$

and of

$$\frac{dt}{ds} = -\frac{d\tau}{ds}\frac{\partial F}{\partial \omega} \qquad [22a]$$

for the group travel time some limiting values need special consideration:

a) $\check{\varepsilon}_o \to 0$ for $\theta \neq 0$ and $\check{\varepsilon}_{\pm 1} \to 0$ for arbitrary θ

b) $\check{\varepsilon}_- \to 0$ for $\theta \neq 0$

c) $\theta \to 0$ for $\check{\varepsilon}_o \neq 0$ and $\check{\varepsilon}_- \equiv \dfrac{\check{\varepsilon}_{+1} - \check{\varepsilon}_{-1}}{2\varepsilon_o} \neq 0$

d) $\theta \to 0$ and $\vec{\varepsilon}_o \to 0$ or $\varepsilon_- \to 0$.

a) The vanishing of one eigenvalue

$$\check{\varepsilon}_s \equiv \varepsilon_o - \sum_c \frac{q_c^2 N_c/m_c\omega}{\omega + i\nu_c + sq_c\, B/\, m_c\, c_o\sqrt{\varepsilon_o\mu_o}} \ (s = 0, \pm 1) \qquad [3]$$

causes the vanishing of one of the two roots $k_\pm^2$ of the dispersion equation (2) (5). With some straightforward calculations we obtain the following limiting values [Rawer - Suchy, 1966 a] for $s\theta \neq 0$:

$$\lim_{\check{\varepsilon}_s \to 0} \zeta = s\,\frac{2\cos\theta}{\sin^2\theta}\,, \quad \lim_{\check{\varepsilon}_s \to 0} \delta = s\,\frac{2\cos\theta}{\sin^2\theta}\,\frac{\varepsilon_o\varepsilon_{-s}}{\varepsilon_o(2\check{\varepsilon}_o - \varepsilon_{-s})} \qquad (27a)$$

$$\lim_{\check\varepsilon_o\to 0}\frac{k_+^2}{\check\varepsilon_o/\varepsilon_o}=\left(\frac{\omega}{c_o}\right)^2\frac{1}{\sin^2\theta}\;,\qquad \lim_{\check\varepsilon_{\pm1}\to 0}\frac{k_-^2}{\check\varepsilon_{\pm1}/\varepsilon_o}=\left(\frac{\omega}{c_o}\right)^2\frac{2}{1+\cos^2\theta}\;. \tag{27b}$$

With this we find from Eq. (25)

$$\lim_{\check\varepsilon_o\to 0}\frac{(ds/d\tau)_+}{\sqrt{\check\varepsilon_o/\varepsilon_o}}=-2\left(\frac{\omega}{c_o}\right)^3\frac{\check\varepsilon_{+1}\,\check\varepsilon_{-1}}{\varepsilon_o^{\;2}}\,\frac{1}{\sin^2\theta} \tag{28a}$$

$$\lim_{\check\varepsilon_{\pm1}\to 0}\frac{(ds/d\tau)_-}{\sqrt{\check\varepsilon_{\pm1}/\varepsilon_o}}=-2\left(\frac{\omega}{c_o}\right)^3\frac{\check\varepsilon_o\,\check\varepsilon_{\mp1}}{\varepsilon_o^2}\sqrt{\frac{1+3\cos^2\theta}{2(1+\cos^2\theta)}}\;. \tag{28b}$$

It is therefore advantageous to write in the vicinity of a zero of $k_\pm$ the integration parameter $d\tau$ as follows:

$$d\tau = ds\,\frac{d\tau}{ds}=\frac{ds}{\sqrt{\check\varepsilon_s/\varepsilon_o}}\left(\sqrt{\frac{\check\varepsilon_s}{\varepsilon_o}}\,\frac{d\tau}{ds}\right) \tag{29}$$

By choosing the intervals ds for the integration proportional to $\sqrt{\check\varepsilon_s/\varepsilon_o}$ the two factors

$$\frac{ds}{\sqrt{\check\varepsilon_s/\varepsilon_o}}\quad\text{and}\quad\sqrt{\frac{\check\varepsilon_s}{\varepsilon_o}}\,\frac{d\tau}{ds}$$

remain finite. At the point $k = 0$ itself

$$d\hat{\vec k} = -\frac{d\tau}{k}\left(\frac{\partial F}{\partial\vec r}-\hat{\vec k}\,\hat{\vec k}\cdot\frac{\partial F}{\partial\vec r}\right) \qquad [20]$$

is infinite, $\vec{k}$ changes through 180°. The limiting values for $d\vec{r}/ds$ are from Eq. (26)

$$\lim_{\check{\varepsilon}_o\to 0}\left(\frac{d\vec{r}}{ds}\right)_+ = \frac{\hat{\vec{k}} - \hat{\vec{B}}\cos\theta}{\sin\theta} \perp \vec{B}\ , \quad \lim_{\check{\varepsilon}_{\pm 1}\to 0}\left(\frac{d\vec{r}}{ds}\right)_- = \frac{\hat{\vec{k}} + \hat{\vec{B}}\cos\theta}{\sqrt{1+3\cos\theta}} \qquad (30)$$

Because $\vec{k}$ and thus θ changes through 180° so does $d\vec{r}/ds$, which means total reflection.

The particular case $\check{\varepsilon}_o \to 0 \leftarrow \theta$ is treated in d) below.

b) If $\check{\varepsilon}_-$ vanishes for finite θ then

$$\zeta \equiv \frac{2\cos\theta}{\sin^2\theta}\ \frac{\check{\varepsilon}_o\check{\varepsilon}_-}{\check{\varepsilon}_{+1}\check{\varepsilon}_{-1} - \check{\varepsilon}_o\check{\varepsilon}_+} \qquad [23b]$$

vanishes, k remains finite, and we obtain from Eqs. (25) and (26) together with Eq. (24a) for δ

$$\lim_{\check{\varepsilon}_-\to 0}\left(\frac{ds}{d\tau}\right)_\pm = \mp 2k_\pm\left(\frac{\omega}{c_o}\right)^2 \sin^2\theta\ \frac{\check{\varepsilon}_+(\check{\varepsilon}_+ - \check{\varepsilon}_o)}{\varepsilon_o^2}\ \sqrt{1+\cot^2\theta\left[1-\left(\frac{c_o}{\omega}k_\pm\right)^2\frac{\varepsilon_o}{\check{\varepsilon}_{+1}}\right]^2} \qquad (31a)$$

$$\lim_{\check{\varepsilon}_-\to 0}\left(\frac{d\vec{r}}{ds}\right)_\pm = \frac{\pm\hat{\vec{k}} + \dfrac{\hat{\vec{k}}\cos\theta - \hat{\vec{B}}}{\sin\theta}\cot\theta\left[1-\left(\dfrac{c_o}{\omega}k_\pm\right)^2\dfrac{\varepsilon_o}{\check{\varepsilon}_{+1}}\right]}{\pm\sqrt{1+\cot^2\theta\left[1-\left(\dfrac{c_o}{\omega}k_\pm\right)^2\dfrac{\varepsilon_o}{\check{\varepsilon}_{+1}}\right]^2}} \qquad (31b)$$

We have no zero of $ds/d\tau$ and the limiting values for $(d\vec{r}/ds)_\pm$ do not allow a simple geometrical representation. Physically the crossing over

$\zeta = 0$ means an alternation in the sense of rotation of the transversal polarization ($\perp \vec{k}$) through linear polarization into the opposite sense of rotation [Stix, 1962]. [For $\check{\varepsilon}_o = 0$ the value $\zeta = 0$ is not crossed over but only touched, because of the total reflection at $\zeta = 0$, see case a) above.] The disappearance of $\check{\varepsilon}_-$ in a plasma is only possible if at least two different species of ions are taken into account, see Eq. (3).

c) The vanishing of $\sin\theta$ causes

$$\zeta \equiv \frac{2\cos\theta}{\sin^2\theta} \frac{\check{\varepsilon}_o \check{\varepsilon}_-}{\check{\varepsilon}_{+1}\check{\varepsilon}_{-1} - \check{\varepsilon}_o \check{\varepsilon}_+} \approx \frac{2\cos\theta}{\sin^2\theta} \frac{1-\tilde{X}}{\tilde{Y}} \qquad [23b]$$

to increase of second order to infinity as long as $\check{\varepsilon}_o$ and $\check{\varepsilon}_-$ remain finite. Therefore we write $(ds/d\tau)_{\pm)}$ Eq. (25), and $(d\vec{r}/ds)_{\pm)}$ Eq. (26), as follows:

$$\left(\frac{ds}{d\tau}\right)_\pm = \mp 2k_\pm \left(\frac{\omega}{c_o}\right)^2 \frac{\check{\varepsilon}_o \check{\varepsilon}_-}{\varepsilon_o^2} 2\cos\theta \,\mathrm{sign}\,(\mathrm{Re}\,\zeta) \sqrt{1 + \frac{1}{\zeta^2} + \frac{\cot^2\theta}{\zeta^2}\left[1 - \left(\frac{c_o}{\omega}k_\pm\right)^2 \frac{\zeta}{\delta}\right]^2} \qquad (32a)$$

$$\frac{d\vec{r}}{ds}_\pm = \frac{\pm\sqrt{1+\zeta^2}\,\sin^2\theta\,\hat{\vec{k}} + \frac{\hat{\vec{k}}\cos\theta - \hat{\vec{B}}}{\sin\theta}\sin\theta\cos\theta\left[1 - \left(\frac{c_o}{\omega}k_\pm\right)^2 \frac{\zeta}{\delta}\right]}{\pm\sqrt{(1+\zeta^2)\sin^4\theta + \sin^2\theta\cos^2\theta\left[1-\left(\frac{c_o}{\omega}k_\pm\right)^2\frac{\zeta}{\delta}\right]^2}} \qquad (32b)$$

The limiting values can now easily be obtained with Eqs. (23b) and (24a) for ζ and δ and Eq. (26a) for the unit vector $(\hat{\vec{k}}\cos\theta - \hat{\vec{B}})/|\sin\theta|$:

$$\lim_{\theta\to 0} \frac{ds}{d\tau}_\pm = \mp 2k_\pm \left(\frac{\omega}{c_o}\right)^2 \frac{\check{\varepsilon}_o \check{\varepsilon}_-}{\varepsilon_o^2} 2\,\mathrm{sign}\,(\mathrm{Re}\,\zeta) \qquad (33a)$$

$$\lim_{\theta\to 0}\left(\frac{d\vec{r}}{ds}\right)_{\pm} = \hat{\vec{k}}, \text{ i.e. } \frac{d\vec{r}}{ds}\,||\vec{B} \text{ for } \vec{k}\,||\,\vec{B} \tag{33b}$$

In this case we have again no zero in $ds/d\tau$. The unit vector (26a) had lead us to a simple limiting value for $d\vec{r}/ds$.

d) If $\sin\theta$ and $\check{\varepsilon}_o$ or $\check{\varepsilon}_-$ vanish simultaneously the corresponding limiting value of

$$\zeta \equiv \frac{2\cos\theta}{\sin^2\theta}\;\frac{\check{\varepsilon}_o\check{\varepsilon}_-}{\check{\varepsilon}_{+1}\check{\varepsilon}_{-1} - \varepsilon_o\varepsilon_+} \tag{23b}$$

may be at first regarded as finite. We therefore write $(ds/d\tau)$, Eq. (25), and $(d\vec{r}/ds)$, Eq. (26), as follows:

$$\left(\frac{ds}{d\tau}\right)_{\pm} = \mp 2k_{\pm}\left(\frac{\omega}{c_o}\right)^2 \sqrt{\frac{\check{\varepsilon}_o\check{\varepsilon}_-}{\varepsilon_o^2}}\;\frac{2\cos\theta}{\zeta}\sqrt{\frac{\check{\varepsilon}_o\check{\varepsilon}_-}{\varepsilon_o^2}(1+\zeta^2) + \frac{\check{\varepsilon}_{+1}\check{\varepsilon}_{-1} - \check{\varepsilon}_o\check{\varepsilon}_+}{\varepsilon_o^2}\;\frac{\zeta}{2}\,|\cos\theta|\left[1-\left(\frac{c_o}{\omega}k_{\pm}\right)^2\frac{\zeta}{\delta}\right]^2} \tag{34a}$$

$$\left(\frac{d\vec{r}}{ds}\right)_{\pm} = \frac{\pm\sqrt{1+\zeta^2}\,\sin\theta\,\hat{\vec{k}} + \frac{\hat{\vec{k}}\cos\theta - \hat{\vec{B}}}{\sin\theta}\,\cos\theta\left[1-\left(\frac{c_o}{\omega}k_{\pm}\right)^2\frac{\zeta}{\delta}\right]}{\pm\sqrt{(1+\zeta^2)\sin^2\theta + \cos^2\theta\left[1-\left(\frac{c_o}{\omega}k_{\pm}\right)^2\frac{\zeta}{\delta}\right]^2}} \tag{34b}$$

This yields the limiting values

$$\lim_{\substack{\theta\to 0\\ \check{\varepsilon}_o\check{\varepsilon}_-\to 0}} \frac{(ds/d\tau)_{\pm}}{\sqrt{\check{\varepsilon}_o\check{\varepsilon}_-/\varepsilon_o^2}} = \mp \lim 2k_{\pm}\left(\frac{\omega}{c_o}\right)^2\frac{2}{\zeta}\sqrt{\frac{\check{\varepsilon}_{+1}\check{\varepsilon}_{-1} - \check{\varepsilon}_o\check{\varepsilon}_+}{\varepsilon_o^2}\;\frac{\zeta}{2}\left[1-\left(\frac{c_o}{\omega}k_{\pm}\right)^2\frac{\zeta}{\delta}\right]^2} \tag{35a}$$

$$\lim_{\substack{\theta\to 0\\ \check{\varepsilon}_o\check{\varepsilon}_-\to 0}} \left(\frac{d\vec{r}}{ds}\right)_\pm = \pm\lim \frac{\hat{\vec{k}}\cos\theta - \hat{\vec{B}}}{\sin\theta} = \pm\lim \frac{(\hat{\vec{B}} + d\hat{\vec{k}}) - \hat{\vec{B}}}{d\theta} = \pm\lim \frac{d\hat{\vec{k}}}{d\theta} \qquad (35b)$$

In Eq. (35a) the limit of $k_\pm$ is finite, because k_+ vanishes for $\varepsilon_o \to 0$ only if $\theta \neq 0$ and k_- remains finite anyway at $\check{\varepsilon}_o = 0$. Analogous to Eq. (29) we write

$$d\tau = ds\,\frac{d\tau}{ds} = \frac{ds}{\sqrt{\check{\varepsilon}_o\check{\varepsilon}_-/\varepsilon_o^2}}\left(\sqrt{\frac{\check{\varepsilon}_o\check{\varepsilon}_-}{\varepsilon_o^2}}\;\frac{d\tau}{ds}\right) \qquad (36)$$

and chose the intervals ds proportional to $\sqrt{\check{\varepsilon}_o\check{\varepsilon}_-/\varepsilon_o^2}$. Then each of the two factors in Eq. (36) remains finite in the vicinity of $\check{\varepsilon}_o\check{\varepsilon}_- = 0$ for $\theta = 0$.

To obtain more information about $\lim(d\vec{r}/ds)$, Eq. (35b), we substitute

$$\frac{d\hat{\vec{k}}}{d\tau} = -\frac{1}{k}\left(\frac{\partial F}{\partial\vec{r}} - \hat{\vec{k}}\,\hat{\vec{k}}\cdot\frac{\partial F}{\partial\vec{r}}\right) \qquad [20]$$

into Eq. (35b) and get with $\hat{\vec{k}} \to \hat{\vec{B}}$

$$\lim\left(\frac{d\vec{r}}{ds}\right)_\pm = \mp\lim\frac{1}{k_\pm}\frac{d\tau}{d\theta}\left(\frac{\partial F}{\partial\vec{r}} - \hat{\vec{B}}\,\hat{\vec{B}}\cdot\frac{\partial F}{\partial\vec{r}}\right) .$$

The factor $\lim(d\tau/kd\theta)$ can be replaced because $d\vec{r}/ds$ is a unit vector:

$$\lim_{\substack{\theta\to 0\\ \check{\varepsilon}_o\check{\varepsilon}_-\to 0}} \frac{d\vec{r}}{ds} = \lim_{\check{\varepsilon}_o\check{\varepsilon}_-\to 0} \frac{\frac{\partial F}{\partial\vec{r}} - \hat{\vec{B}}\,\hat{\vec{B}}\cdot\frac{\partial F}{\partial\vec{r}}}{\sqrt{\frac{\partial F}{\partial\vec{r}}\cdot\frac{\partial F}{\partial\vec{r}} - \left(\hat{\vec{B}}\cdot\frac{\partial F}{\partial\vec{r}}\right)^2}} = \lim_{\check{\varepsilon}_o\check{\varepsilon}_-\to 0} \frac{\left(\hat{\vec{B}}\times\frac{\partial F}{\partial\vec{r}}\right)\times\hat{\vec{B}}}{\left|\left(\hat{\vec{B}}\times\frac{\partial F}{\partial\vec{r}}\right)\times\hat{\vec{B}}\right|} . \qquad (37)$$

Thus $\lim d\vec{r}/ds$ is perpendicular to $\vec{B}$ and in the plane through the vectors $\vec{B}$ and $\partial F/\partial\vec{r}$.

The sign of the root in Eq. (37) is still undertermined. To decide whether it changes or not during the transition $\theta \to 0 \leftarrow \check{\varepsilon}_o\check{\varepsilon}_-$ one has to distinguish the cases $\check{\varepsilon}_o \to 0$ and $\check{\varepsilon}_- \to 0$. A change would mean total reflection.

For $\varepsilon_- \to 0$ we can obtain the limiting value (35b) for $d\vec{r}/ds$ also from Eq. (31b), with $\theta \to 0$. Hence $d\vec{r}/ds$ does not show a peculiar behavior in this case and we cannot await total reflection.

For $\check{\varepsilon}_o \to 0$ we obtain the same limiting values for $(d\vec{r}/ds)_+$ from Eq. (35b) and from Eq. (30). The latter holds, however, for a case of total reflection and thus we have total reflection also in Eq. (37) for the +-mode. Since k_+ remains finite for $\theta = 0$ we have, according to Eq. (20), no sudden change of the direction $\hat{\vec{k}}$ through 180^o in this case. This causes a cusp in the ray path [Poeverlein 1949, 1950].

If

$$\lim_{\substack{\theta\to 0\\ \check{\varepsilon}_o\check{\varepsilon}_-\to 0}} \zeta \equiv \lim_{\substack{\theta\to 0\\ \check{\varepsilon}_o\check{\varepsilon}_-\to 0}} \frac{2\cos\theta}{\sin^2\theta}\ \frac{\check{\varepsilon}_o\check{\varepsilon}_-}{\check{\varepsilon}_{+1}\check{\varepsilon}_{-1} - \check{\varepsilon}_o\check{\varepsilon}_+} = \infty$$

the results depend on the sequence of the limiting processes. The limit of the cusps for the +-mode would be a kink. But the two modes are strongly coupled in this particular case and a special discussion is needed [Poeverlein 1949, 1950].

7. The group speed in a gyrotropic incompressible plasma

The expression $\partial F/\partial\omega$ for the reciprocal group speed, Eq. (22a), is rather involved, since the eigenvalues

$$\frac{\check{\varepsilon}_s}{\varepsilon_o} = 1 - \sum_c \frac{\omega_{Nc}^2/\omega}{\omega + i\nu_c + s\omega_{Bc}} \quad \text{with} \quad \omega_{Nc}^2 \equiv \frac{q_c^2 N_c}{\varepsilon_o m_c}, \quad \omega_{Bc} \equiv \frac{q_c B}{m_c c_o \sqrt{\varepsilon_o \mu_o}} \qquad [3]$$

in the coefficients (5) of the dispersion equation

$$F \equiv k^4 A - \left(\frac{\omega}{c_o}\right)^2 k^2 B + \left(\frac{\omega}{c_o}\right)^4 C = 0 \qquad [2]$$

depend on ω in a somewhat complicated manner. It is advantageous to write

$$\frac{\partial F}{\partial\omega} = \frac{2}{\omega}\frac{\partial\omega^2 F}{\partial\omega^2} - \frac{2F}{\omega} \equiv \frac{2}{\omega}(f - \beta F) \qquad (38)$$

with the quadratic polynomial in k^2

$$f \equiv k^4 a - \left(\frac{\omega}{c_o}\right)^2 k^2 b + \left(\frac{\omega}{c_o}\right)^4 c \qquad (38a)$$

having the (dimensionless) coefficients

$$a = \frac{\partial\omega^2 A}{\partial\omega^2} \qquad b = \frac{1}{\omega^2}\frac{\partial\omega^4 B}{\partial\omega^2} \qquad c = \frac{1}{\omega^4}\frac{\partial\omega^6 C}{\partial\omega^2}\,. \qquad (38b)$$

The (finite) factor β is arbitrary because $F = 0$.

With the eigenvalues

$$\alpha_s \equiv \frac{\partial}{\partial\omega^2}\left(\frac{\omega^2\check{\varepsilon}_s}{\varepsilon_o}\right) \quad \text{of the tensor} \quad \overleftrightarrow{\alpha} \equiv \frac{\partial}{\partial\omega^2}\left(\frac{\omega^2\overleftrightarrow{\varepsilon}}{\varepsilon_o}\right) \tag{39}$$

we can write the coefficients a, b, c, Eq. (38b), using the expressions (5) for A, B, C:

$$a = \alpha_+ + (\alpha_o - \alpha_+)\cos^2\theta$$

$$b = \left(\alpha_o \frac{\check{\varepsilon}_+}{\varepsilon_o} + \frac{\check{\varepsilon}_o}{\varepsilon_o}\alpha_+\right)(1 + \cos^2\theta) + \left(\alpha_{+1}\frac{\check{\varepsilon}_{-1}}{\varepsilon_o} + \alpha_{-1}\frac{\check{\varepsilon}_{+1}}{\varepsilon_o}\right)(1 - \cos^2\theta)$$

$$c = \alpha_o \frac{\check{\varepsilon}_{+1}\check{\varepsilon}_{-1}}{\varepsilon_o^2} + \alpha_{+1}\frac{\check{\varepsilon}_o\check{\varepsilon}_{-1}}{\varepsilon_o^2} + \alpha_{-1}\frac{\check{\varepsilon}_o\check{\varepsilon}_{+1}}{\varepsilon_o^2} \; . \tag{40}$$

The eigenvalues α_s and their linear combination $\alpha_+ \equiv \frac{1}{2}(\alpha_{+1} + \alpha_{-1})$ can easily be calcuated with Eq. (3), yielding

$$\alpha_s = 1 - \frac{1}{2}\sum_c \frac{\frac{\omega_{Nc}^2}{\omega^2}\left(s\frac{\omega_{Bc}}{\omega} + i\frac{\nu_c}{\omega}\right)}{\left(1 + i\frac{\nu_c}{\omega} + s\frac{\omega_{Bc}}{\omega}\right)^2} \tag{41a}$$

$$\alpha_+ = \tag{41b}$$

$$= 1 + \sum_c \frac{\omega_{Nc}^2}{\omega^2}\,\frac{\left(\frac{\omega_{Bc}}{\omega}\right)^2\left(1 + i\frac{\nu_c}{\omega}\right) - \frac{i}{2}\frac{\nu_c}{\omega}\left[\left(1 + i\frac{\nu_c}{\omega}\right)^2 + \left(\frac{\omega_{Bc}}{\omega}\right)^2\right]}{\left(1 + i\frac{\nu_c}{\omega}\right)^2 - \left(\frac{\omega_{Bc}}{\omega}\right)^2}$$

For the determination of the group travel time we have with Eqs. (22a) (38)

$$dt = -d\tau \frac{\partial F}{\partial \omega} = -ds \frac{d\tau}{ds} \frac{2}{\omega} (f - \beta F)$$

with $d\tau/ds$ given by Eq. (25), the polynomials f and F given by Eqs. (38a,b) (40) (41a,b) and (2) (3) (5). The factor β is arbitrary and may be chosen as zero.

8. The second Hamilton equation in curvilinear coordinates

Our last task is the calculation of the spatial derivative $\partial F/\partial \vec{r}$ in the second Hamilton equation

$$\frac{d\vec{k}}{d\tau} = -\left(\frac{\partial F}{\partial \vec{r}}\right)_{\vec{k}} \quad . \qquad [14a]$$

For this partial differentiation the vector

$$\vec{k} = k^{\nu}\vec{g}_{\nu} = k_{\nu}\vec{g}^{\nu} \quad \text{(summation over repeated suffixes)} \qquad (42)$$

has to be kept constant. In cartesian coordinates with constant basis vectors $\vec{g}_{\nu} = \vec{g}^{\nu}$ this means constant components $k^{\nu} = k_{\nu}$. In general curvilinear coordinates, however, the condition is the vanishing of

$$d\vec{k} = d(k^{\nu}\vec{g}_{\nu}) = dk^{\nu}\vec{g}_{\nu} + k^{\nu}d\vec{g}_{\nu} = dk^{\nu}\vec{g}_{\nu} + k^{\nu}d\vec{r} \cdot \vec{g}^{\mu} [\nu\mu,\lambda] \vec{g}^{\lambda} \qquad (43)$$

[Fues 1957] with the C hristoffel symbol of first kind

$$[\nu\mu,\lambda] \equiv \frac{1}{2}\left(\frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} + \frac{\partial g_{\nu\lambda}}{\partial x^{\mu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\lambda}}\right) . \tag{43a}$$

Therefore we have to write

$$\left(\frac{\partial F}{\partial \vec{r}}\right)_{\vec{k}} = \left(\frac{\partial F}{\partial \vec{r}}\right)_{k^{\nu}} + \left(\frac{\partial k^{\nu}}{\partial \vec{r}}\right)_{\vec{k}} \frac{\partial F}{\partial k^{\nu}} = \left(\frac{\partial F}{\partial \vec{r}}\right)_{k^{\nu}} + \left(\frac{\partial k^{\nu}}{\partial \vec{r}}\right)_{\vec{k}} \vec{g}_{\nu} \cdot \frac{\partial F}{\partial \vec{k}}$$

Putting $d\vec{k} = 0$ we find with Eq. (43)

$$\left(\frac{\partial k^{\nu}}{\partial \vec{r}}\right)_{\vec{k}} \vec{g}_{\nu} = - k^{\nu}\vec{g}^{\mu} [\nu\mu,\lambda] \vec{g}^{\lambda} ,$$

hence

$$\frac{\partial F}{\partial \vec{r}}_{\vec{k}} = \left(\frac{\partial F}{\partial \vec{r}}\right)_{k^{\nu}} - k^{\nu}\vec{g}^{\mu} [\nu\mu,\lambda] \vec{g}^{\lambda} \cdot \frac{\partial F}{\partial \vec{k}} . \tag{44}$$

Substituting $d\vec{k}/d\tau$ from Eq. (43) into the left-hand side of the second Hamilton equation (14a) and Eq. (44) into its right-hand side we obtain with $d\vec{r}/d\tau = \partial F/\partial \vec{k}$ (14a)

$$\frac{dk^{\nu}}{d\tau}\vec{g}_{\nu} + \frac{\partial F}{\partial \vec{k}} \cdot k^{\nu}\vec{g}^{\mu}[\nu\mu,\lambda]\vec{g}^{\lambda} = -\left(\frac{\partial F}{\partial \vec{r}}\right)_{k^{\nu}} + \frac{\partial F}{\partial \vec{k}} \cdot k^{\nu}\vec{g}^{\mu}[\nu\lambda,\mu]\vec{g}^{\lambda} ,$$

and with

$$[\nu\mu,\lambda] - [\nu\lambda,\mu] = \frac{\partial g_{\nu\lambda}}{\partial x^{\mu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\lambda}} \qquad [43a]$$

we have finally

$$\frac{dk^{\nu}}{d\tau}\vec{g}_{\nu} = -\left(\frac{\partial F}{\partial \vec{r}}\right)_{k^{\nu}} + \frac{\partial F}{\partial \vec{k}} \cdot k^{\nu}\vec{g}^{\mu}\left(\frac{\partial g_{\nu\mu}}{\partial x^{\lambda}} - \frac{\partial g_{\nu\lambda}}{\partial x^{\mu}}\right)\vec{g}^{\lambda}. \qquad (45)$$

To eliminate the basis vectors we use

$$\frac{\partial F}{\partial \vec{r}} = \vec{g}^{\lambda}\frac{\partial F}{\partial x^{\lambda}}, \quad \frac{\partial F}{\partial \vec{k}} = \vec{g}^{l}\frac{\partial F}{\partial k^{l}}$$

$$\vec{g}_{\nu}\cdot\vec{g}^{\kappa} = \delta^{\kappa}_{\nu}, \quad \vec{g}^{\lambda}\cdot\vec{g}^{\kappa} = g^{\lambda\kappa}$$

for the multiplication of Eq. (45) with $\vec{g}^{\kappa}$, yielding [Haselgrove 1955]

$$\frac{dk^{\kappa}}{d\tau} = -g^{\kappa\lambda}\frac{\partial F}{\partial x^{\lambda}} + \frac{\partial F}{\partial k^{l}}k^{\nu}g^{l\mu}\left(\frac{\partial g_{\nu\mu}}{\partial x^{\lambda}} - \frac{\partial g_{\nu\lambda}}{\partial x^{\mu}}\right)g^{\lambda\kappa}$$

If the curvilinear coordinates are orthogonal we can transform the contravariant components k^{κ} into the "physical components"

$$k_{(\kappa)} \equiv g_{\kappa}k^{\kappa} \quad \text{with} \quad g_{\kappa} \equiv \sqrt{g_{\kappa\kappa}} = 1/\sqrt{g^{\kappa\kappa}} \quad \text{(no summation)}$$

which has been done by Haselgrove [1955].

The equation

$$\frac{d\vec{k}}{d\tau} = -\frac{1}{k}\left[\left(\frac{\partial F}{\partial \vec{r}}\right)_{\vec{k}} - \hat{\vec{k}}\,\hat{\vec{k}} \cdot \left(\frac{\partial F}{\partial \vec{r}}\right)_{\vec{k}}\right] \quad [20]$$

is very involved in curvilinear coordinates, because the condition of $\vec{k}$ being a unit vector complicates seriously all expressions, even for orthogonal coordinates.

9. Concluding remarks

In our deduction of Hamilton's equations we have intentionally avoided as starting point an integral principle such as Fermat's principle, since the form of its integrand L (the "Lagrangean") and its integration variable $d\tau$ cannot be deduced from physical assumptions, but must be justified a posteriori. On the other hand, from Hamilton's equations (14a, b) the following integral principle can be derived [Rawer-Suchy 1966b]:

$$\delta \int_{\vec{r}_o, t_o}^{\vec{r}, t} d\tau \; L(\vec{k}, \omega; \vec{r}, t) = 0 \quad \text{with} \quad \delta\vec{r}_o = \delta t_o = 0 = \delta\vec{r} = \delta t$$

with

$$d\tau L = d\tau \left(\frac{\partial F}{\partial \vec{k}} \cdot \vec{k} + \frac{\partial F}{\partial \omega}\, \omega\right) = dt\left(\frac{d\vec{r}}{dt} \cdot \vec{k} - \omega\right).$$

For constant ω this yields Fermat's principle

$$\delta \int_{\vec{r}_o}^{\vec{r}} d\vec{r} \cdot \vec{k} = 0 \;.$$

The validity of Hamilton's equations is only restricted by the applicability of geometric optics for the problem under consideration and the weakness of absorption. To omit the latter restriction some proposals have been made [Brandstatter 1963; Allis, Buchsbaum, Bers 1963] but they seem to be controversial [Furutsu 1952].

If geometric optics fails to provide a reasonable approximation full wave optics has to be employed. Among anisotropic media only "Stratified" media have hitherto been treated thoroughly (a survey is given by Rawer and Suchy [1966c]). On the other hand, for these stratified anisotropic media equations for ray tracing and group speed have been deduced using the method of stationary phase [Booker 1939; Millington 1951]. They can be shown to be special cases of Hamilton's equations [Rawer - Suchy 1966d].

The full strength of Hamilton's equations becomes evident in applications to non-stratified anisotropic media, e.g. the magnetosphere and also the ionosphere with large local density irregularities. For the latter numerical computations of group paths and travel times with Hamilton's equations are now being performed at the Institute for Telecommunication Sciences and Aeronomy, Environmental Science Services Administration, Boulder, Colorado [Paul, Smith and Wright 1965].

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